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The Pauli algebra approach to special relativity

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Abstract. The Pauli algebra $\mathscr{P}$, in which the usual dot and cross products of 3-space vectors are combined in an associative, invertible, but non-commutative multiplication, provides a simple but powerful approach to problems in special relativity. Even though the Pauli algebra is the Clifford algebra for Euclidean 3-space, Minkowski 4-vectors and their products in the Minkowski metric appear in a natural and covariant way as elements of $\mathscr{P}$. We review the algebra and develop a formulation which, although closely tied to elementary vector and functional analysis, nevertheless allows a compact coordinate-free treatment of essentially all problems in special relativity. We derive a number of useful results and show how the elements are related both to traditional Minkowski-space tensors and to elements of the Dirac algebra.

1. Introduction

Clifford algebras have become popular tools in the description of relativistic physics (Hestenes 1966, Choquet-Bruhat *et al* 1977, Salingaros and Dresden 1983, Chisholm and Common 1986). They provide a compact coordinate-free description of special relativity. The elements of such algebras are all the 'multivectors' which can be formed from linear combinations of antisymmetric products of a set of basis vectors. For example, starting with the four basis vectors of Minkowski space, one constructs the Dirac algebra D from sixteen possible basis forms: one scalar, four vectors, six bivectors, four pseudovectors and one pseudoscalar. The name 'Dirac algebra' comes from the well known representation of its basis vectors by the Dirac matrices of relativistic quantum theory. It seems the natural choice for problems of special relativity and has been applied by a number of workers (Hestenes 1974, Hamilton 1981, Grieder 1984, Salingaros 1985, 1986). However D, or its complexification with 32 basis forms, contains element types and relations which are unfamiliar to most physicists, and perhaps as a result, it has not been widely adopted by the physics community.

The simpler Clifford algebra built on the three basis vectors of three-dimensional Euclidean space is known as the Pauli algebra $\mathscr{P}$, named after the representation of its basis vectors as Pauli spin matrices. The basis vectors can also be represented by complex quaternions. The Pauli algebra in one or another of its explicit representations has long been known to be useful for many problems in special relativity (Klein and Sommerfeld 1910, Weiss 1941, MacFarlane 1962, Misner *et al* 1973, Frost 1975, Baylis 1980, Baylis and Jones 1988). Like the larger Clifford algebras, it offers an economical coordinate-free formulation of problems. Unlike them, its elements can all be written as complex linear combinations of scalars and vectors, and its associative invertible algebraic multiplication is trivially expressed in terms of the familiar dot and cross products of vectors. Minkowski 4-vectors appear naturally as elements of $\mathscr{P}$, where they consist of a real scalar (the time component) plus a real vector. We have recently

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argued that $\mathscr{P}$ should be as powerful as D in practically all applications in special relativity (Baylis and Jones 1988). The argument can be summarised as follows.

The sixteen basis forms of D can be divided into eight even forms (the scalar, six bivectors and a pseudoscalar) and eight odd ones (the vectors and pseudovectors). The even multivectors of D, which can be written as linear combinations of the even basis forms, constitute an algebra in themselves, the even Dirac algebra D_+, which is isomorphic to $\mathscr{P}$. The odd multivectors of D do not constitute an algebra since the product of two odd multivectors is even. Nevertheless, they can be mapped onto $\mathscr{P}$ by multiplying each of them by one of the basis vectors, thereby changing them to even multivectors. Since physically interesting quantities always seem to be either even or odd multivectors and not mixtures, the double mapping of D onto the half-size $\mathscr{P}$ does not restrict the latter's power in treating physical systems.

Many of the results given here have been published previously, often not in terms of $\mathscr{P}$ but in the language of the isomorphic algebras of complex quaternions or of 2×2 complex matrices in terms of higher-order Clifford algebras. However, in order to facilitate future applications it is important to collect the results in a single unified notation in which the coordinate-free matrix-free compactness of Clifford algebras makes a minimal extension of familiar vector and functional analysis. Results which we believe are new include the use of eigenprojectors of $\mathscr{P}$-elements to evaluate arbitrary functions (§ 3), relations among scalar products including derivatives (§ 3), the explicit connection of some $\mathscr{P}$-elements to higher-rank Minkowski tensors and their duals (§ 5), and relations involving bispinnors in $\mathscr{P}$ (§ 6). A following paper (Baylis and Jones 1989) applies the formulation to the relativistic dynamics of charges in external fields.

In § 2, we review the algebraic multiplication of vectors and show how the existence of an associative invertible vector product gives $\mathscr{P}$ much of the facility with vectors that the familiar complex analysis has with complex numbers. In particular, simple expressions in $\mathscr{P}$ are found for analytic functions of vectors. The rotation operator is derived from a pair of reflections and is associated with the group SU(2).

In § 3, the results are extended to general elements of $\mathscr{P}$, which have both scalar and vector parts. The scalar product required to establish a multiplicative inverse embodies the Minkowski metric, and transformations which leave it invariant define the homogeneous Lorentz group. Eigenvalues and eigenprojectors of $\mathscr{P}$-elements are introduced in order to express functions of $\mathscr{P}$-elements.

In § 4, physical Lorentz 4-vectors are associated with real elements of $\mathscr{P}$, and their Lorentz transformations are found in terms of multiplication and addition within $\mathscr{P}$. $\mathscr{P}$-elements of the restricted Lorentz group are exponentials of a complex vector, the six independent components of which give the rotation and boost parameters, and the 2×2 matrices representing these $\mathscr{P}$-elements form the group SL(2, C). Transformation laws for products of 4-vectors are easily found. They can be grouped into two classes: odd products transform like 4-vectors whereas even products transform like 6-vectors, which are equivalent to antisymmetric second-rank tensors in Minkowski space. Examples show how higher-rank objects can also appear in $\mathscr{P}$.

The formal relationship of $\mathscr{P}$-elements and Minkowski tensors is given explicitly in § 5. By expanding $\mathscr{P}$-elements in the four basis elements σ_μ, the traditional Minkowski components are obtained directly.

The paper concludes in § 6 with a brief discussion of spinors and bispinors in $\mathscr{P}$. The Dirac equation is shown to take a simple form which directly relates the various representations by rotations in two two-dimensional spaces.

2. Vectors and their products

The Pauli algebra $\mathscr{P}$ is the Clifford algebra $A^{3,0}$ of R^3 (Choquet-Bruhat *et al* 1977). Its elements are all linear combinations of scalars and 3-space vectors. The algebraic multiplication of vectors is defined to have properties usually associated with the multiplication of square matrices. In this way a matrix representation of the algebra is anticipated. Thus the product of vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ is associative:

$$(\boldsymbol{xy})\boldsymbol{z} = \boldsymbol{x}(\boldsymbol{yz}) \equiv \boldsymbol{xyz} \tag{1}$$

but not generally commutative, and the complex conjugation of a product $\boldsymbol{xy}$ is

$$(\boldsymbol{xy})^+ = \boldsymbol{y}^+\boldsymbol{x}^+ \tag{2}$$

(the symbol for Hermitian conjugation is used in order to make the algebra and its matrix representations formally similar).

Of course, any product can be split into commuting and anticommuting parts:

$$\boldsymbol{xy} = \boldsymbol{x}\cdot\boldsymbol{y} + \mathrm{i}\boldsymbol{x}\times\boldsymbol{y} \tag{3}$$

where the dot and cross products are defined by

$$\boldsymbol{x}\cdot\boldsymbol{y} = \tfrac{1}{2}(\boldsymbol{xy}+\boldsymbol{yx}) \tag{4a}$$

$$\boldsymbol{x}\times\boldsymbol{y} = \frac{1}{2\mathrm{i}}(\boldsymbol{xy}-\boldsymbol{yx}) \equiv \frac{1}{2\mathrm{i}}[\boldsymbol{x}, \boldsymbol{y}]. \tag{4b}$$

The appearance of the imaginary i in (3) ensures that, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are real, then both $\boldsymbol{x}\cdot\boldsymbol{y}$ and $\boldsymbol{x}\times\boldsymbol{y}$ are real as well (see (2)). The dot and cross products so defined turn out to have precisely their usual meanings. The dot product is a scalar, invariant under rotations, and like other (possibly complex) scalars commutes with all other elements of $\mathscr{P}$. The cross product, on the other hand, behaves as a vector under rotations but is invariant under a spatial inversion which changes the signs of both $\boldsymbol{x}$ and $\boldsymbol{y}$: it is called a pseudovector and is easily identified in $\mathscr{P}$ as an imaginary vector.

Much of the beauty and simplicity of the Pauli algebra lies in the fact that the algebraic product (3) is *associative* and *invertible* (as long as the factors are not null) even though neither the dot product nor the cross product have these properties separately. Neither the dot nor the cross product individually contains enough information to allow one of the vector factors to be determined if the other is known, but when combined in the algebraic product (3) there *is* enough information, and one can invert (3) to give

$$\boldsymbol{x} = (\boldsymbol{xy})\boldsymbol{y}/\boldsymbol{y}^2 \tag{5}$$

where $\boldsymbol{y}^2$ is a scalar equal to the squared length of the vector. Note from (3) that vectors are parallel iff (if and only if) they commute and they are perpendicular iff they anticommute.

Any vector can be expanded in a basis of orthogonal unit vectors $\{\sigma_1, \sigma_2, \sigma_3\}$ and these basis vectors must obey the same multiplication rules (3) as other vectors. The Pauli spin matrices obey such rules and can thus be used to represent the basis vectors and thereby to generate a 2×2 matrix representation of the algebra. (The scalars of $\mathscr{P}$ are multiplied by the unit matrix in order to complete the representation.) The expansion coefficients of the σ_i are the components of the vectors on an orthogonal system of coordinates. However, one of the strengths of Clifford algebras is that all calculations can be performed without explicit reference to a specific system of coordinates or to the corresponding components.

Higher-order products of vectors are easily found by iterating (3). Thus one finds that products of an odd number of vectors are generally vectors plus pseudoscalars (which change sign under inversion and appear in $\mathscr{P}$ as imaginary scalars), whereas products of an even number of vectors are scalars plus pseudovectors. Just as the pseudovector part of $\boldsymbol{xy}$ (3) gives the directed area of the parallelogram generated by $\boldsymbol{x}$ and $\boldsymbol{y}$, so the pseudoscalar part of $\boldsymbol{xyz}$ gives

$$\frac{1}{2\mathrm{i}}(\boldsymbol{xyz}-\boldsymbol{zyx})=\boldsymbol{x}\times\boldsymbol{y}\cdot\boldsymbol{z} \tag{6}$$

the volume of the parallelepiped generated by the three vectors. (Where different types of vector products are combined, we establish the following hierarchy in order to reduce the number of parentheses required: first evaluate cross products, then dot products, and finally algebraic products.)

As a simple application, we use (4) to express the triple cross product $\boldsymbol{x}\times(\boldsymbol{y}\times\boldsymbol{z})$ in terms of algebraic products, and then—by adding and subtracting both $\boldsymbol{zxy}$ and $\boldsymbol{yxz}$ and grouping symmetric products (see (4a)) together—in terms of dot products:

$$\begin{aligned}\boldsymbol{x}\times(\boldsymbol{y}\times\boldsymbol{z}) &= -\tfrac{1}{4}[\boldsymbol{x},[\boldsymbol{y},\boldsymbol{z}]]\\ &= -\tfrac{1}{4}[\boldsymbol{xyz}-\boldsymbol{xzy}-\boldsymbol{yzx}+\boldsymbol{zyx}]\\ &= \boldsymbol{yx}\cdot\boldsymbol{z}-\boldsymbol{x}\cdot\boldsymbol{yz}\end{aligned} \tag{7}$$

all without appealing to components or invoking the linear dependence of vectors.

(Treatments of Clifford algebras (Hestenes 1966, Salingaros and Dresden 1983) often introduce an antisymmetric outer (or 'Grassmann') product, indicated by the wedge symbol $\wedge$. In $\mathscr{P}$, these outer products can be identified by

$$\begin{aligned}\boldsymbol{x}\wedge\boldsymbol{y} &= \tfrac{1}{2}[\boldsymbol{x},\boldsymbol{y}]=\mathrm{i}\boldsymbol{x}\times\boldsymbol{y}\\ \boldsymbol{x}\wedge\boldsymbol{y}\wedge\boldsymbol{z} &= (\boldsymbol{x}\wedge\boldsymbol{y})\wedge\boldsymbol{z}=\boldsymbol{x}\wedge(\boldsymbol{y}\wedge\boldsymbol{z})\\ &= \tfrac{1}{2}[(\boldsymbol{x}\wedge\boldsymbol{y})\boldsymbol{z}+\boldsymbol{z}(\boldsymbol{x}\wedge\boldsymbol{y})]=\mathrm{i}(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{z}\end{aligned}$$

and the imaginary element i itself arises from the outer product of the three basis vectors: $\mathrm{i}=\sigma_1\wedge\sigma_2\wedge\sigma_3=\sigma_1\sigma_2\sigma_3$. In our formulation of the Pauli algebra, we avoid use of the outer product by employing the more familiar real dot and cross products of vectors.)

Multiple products of a vector with itself are particularly simple:

$$\boldsymbol{x}^n=\begin{cases}x^n & n \text{ even}\\ x^n\hat{\boldsymbol{x}} & n \text{ odd}\end{cases} \tag{8}$$

where here $x=(\boldsymbol{x}\cdot\boldsymbol{x})^{1/2}$ is the length of the vector and $\hat{\boldsymbol{x}}=\boldsymbol{x}/x$ is its direction. Any analytic function $f(x)$ of a scalar variable is easily extended to a function of a vector by substituting (8) into the power series expansion. The resulting function $f(\boldsymbol{x})$ can be decomposed into an even scalar part and an odd vector part parallel to $\boldsymbol{x}$:

$$f(\boldsymbol{x})=f_+(x)+f_-(x)\hat{\boldsymbol{x}} \tag{9}$$

where $f_\pm(x)=[f(x)\pm f(-x)]/2$. For example

$$\begin{aligned}\exp(\boldsymbol{x}) &= \cosh\boldsymbol{x}+\sinh\boldsymbol{x}\\ &= \cosh x+\hat{\boldsymbol{x}}\sinh x.\end{aligned} \tag{10}$$

Note that the function of a vector variable commutes with that variable.

Many products of vectors are simplified by decomposing one of the vectors into parts parallel and perpendicular to another. For example, in the product $\hat{n}x\hat{n}$, where $\hat{n}$ is any real unit vector, one splits x into parts parallel ($x_{\parallel}$) and perpendicular ($x_{\perp}$) to $\hat{n}$. Since parallel vectors commute and perpendicular ones anticommute,

$$\hat{n}x\hat{n} = \hat{n}x_{\parallel}\hat{n} + \hat{n}x_{\perp}\hat{n} = (x_{\parallel} - x_{\perp})\hat{n}\hat{n} = x_{\parallel} - x_{\perp}. \tag{11}$$

Consequently, the transformation

$$x \to -\hat{n}x^{+}\hat{n} \tag{12}$$

is a *reflection* of x in the plane normal to $\hat{n}$. (The conjugation ensures that pseudovectors are reflected properly.) Reflections are easily compounded: a reflection in $\hat{n}_1$ followed by a reflection in $\hat{n}_2$, is given by

$$x \to \hat{n}_2\hat{n}_1 x \hat{n}_1\hat{n}_2. \tag{13}$$

By (3) we can write

$$\begin{aligned}\hat{n}_1\hat{n}_2 &= \hat{n}_1 \cdot \hat{n}_2 + \mathrm{i}\hat{n}_1 \times \hat{n}_2\\ &= \cos\theta/2 + \mathrm{i}\hat{\theta}\sin\theta/2 = \exp(\mathrm{i}\boldsymbol{\theta}/2)\end{aligned} \tag{14}$$

where $\theta/2$ is the angle of the opening between the two planes and $\hat{\theta}$ lies along their line of intersection. Splitting x into parts $x_{\parallel} = x \cdot \hat{\theta}\hat{\theta}$ parallel and $x_{\perp} = x - x_{\parallel}$ perpendicular to $\hat{\theta}$, the double reflection (13) is seen to be a rotation of x by $\boldsymbol{\theta}$:

$$\begin{aligned}x \to \exp(-\mathrm{i}\boldsymbol{\theta}/2)x\exp(\mathrm{i}\boldsymbol{\theta}/2) &= x_{\parallel} + \exp(-\mathrm{i}\boldsymbol{\theta})x_{\perp}\\ &= x_{\parallel} + x_{\perp}\cos\theta + \hat{\theta} \times x\sin\theta.\end{aligned} \tag{15}$$

It is easily seen that rotations leave the scalar x^2 invariant. Matrix representations of the rotation operators form the group SU(2) of unimodular unitary 2×2 matrices. We note in (15) that the two operators $\pm\exp(-\mathrm{i}\boldsymbol{\theta}/2)$ correspond to the same physical rotation. Furthermore, a unit vector $\hat{n}$ may be viewed as i times the rotation operator $\exp(-\mathrm{i}\pi\hat{n}/2)$ for a rotation of 180° about $\hat{n}$, and thus the general rotation ((13) and (15)) may be obtained from two 180° rotations as well as from two reflections.

3. General elements of $\mathscr{P}$

From the algebraic multiplication of vectors discussed above, one can see that the most general element of $\mathscr{P}$ is the sum of a complex scalar, say a_0, and a complex vector $\boldsymbol{a}$:

$$a = a_0 + \boldsymbol{a}. \tag{16}$$

The product ab of two elements a and $b = b_0 + \boldsymbol{b}$ is found by combining the rule (3) for vector multiplication with the usual multiplication by scalars. The dot product of any two elements a and b is defined as the scalar

$$a \cdot b = a_0 b_0 + \boldsymbol{a} \cdot \boldsymbol{b} \tag{17}$$

from which it follows that $a \cdot b = b \cdot a$, $1 \cdot a$ is the scalar part of a and is identical with one-half the trace of the 2×2 matrix representing a, and $1 \cdot (ab) = a \cdot b = 1 \cdot (ba)$. Of course the algebraic multiplication of $\mathscr{P}$-elements is associative:

$$a(bc) = (ab)c \equiv abc \tag{18}$$

so that the scalar parts

$$1 \cdot (abc) = a \cdot (bc) = (ab) \cdot c = 1 \cdot (bca) \tag{19}$$

may be calculated in several equivalent ways.

In the previous section, the inverse of vector multiplication could be established because the square of a vector is a scalar. We want to extend the result to multiplication of general elements of $\mathcal{P}$, but $a^2 = a \cdot a + 2a_0\boldsymbol{a}$ is not a scalar unless either the scalar or vector part of a vanishes. However, an appropriate scalar product does exist. To every $a = a_0 + \boldsymbol{a} \in \mathcal{P}$, there exists a related element $\bar{a}$ obtained by changing the sign of the vector part:

$$\bar{a} = a_0 - \boldsymbol{a}$$

which we call the 'spatial reverse' of a. Note that the spatial reverse of $\bar{a}$ is a and that the sum

$$a + \bar{a} = 2 \cdot a = 2a_0$$

is a scalar. Indeed a given element of $\mathcal{P}$ is a scalar iff it is equal to its spatial reverse. The spatial reverse of a product is easily established from (3):

$$\overline{ab} = \bar{b}\bar{a}. \tag{20}$$

Since $a\bar{a}$ is its own spatial reverse, the product of any element with its spatial reverse must be a scalar. It is called the 'modulus' of a:

$$\begin{aligned} a\bar{a} &= a \cdot \bar{a} = 1 \cdot (a\bar{a}) = \bar{a} \cdot a = \bar{a}a \\ &= a_0^2 - \boldsymbol{a}^2 \end{aligned} \tag{21}$$

and is identical to the determinants of the matrix representation both of a and of $\bar{a}$. Any element whose modulus vanishes is said to be a *null* element. If its modulus is unitary, it is *unimodular.*

Any non-null element a has a multiplicative inverse

$$a^{-1} = \bar{a}/(a \cdot \bar{a}). \tag{22}$$

The inverse of a unimodular element is simply its spatial reverse. It is remarkable that the scalar of (21) needed to form multiplicative inverses in $\mathcal{P}$ suggests the Minkowski metric of special relativity even though $\mathcal{P}$ is the Clifford algebra for Euclidean 3-space. Evidently the scalar part of some $\mathcal{P}$-elements may be interpreted as the zeroth or time components of Lorentz 4-vectors. If, in analogy with rotations, we look for transformations which leave the scalar $a\bar{a}$ invariant, we obtain the Lorentz group. Explicit expressions for the transformations will be found in § 4. Anticipating a Minkowski-space interpretation, we define two elements a and b to be *orthogonal* when the scalar $a \cdot \bar{b} = b \cdot \bar{a}$ vanishes. When they anticommute, they are perpendicular 3-vectors (a special class of orthogonal elements).

In (9) we found a simple expression for a function of a vector. To extend this result to the function of a general element of $\mathcal{P}$, we need an expression for powers a^n of a. Since $a + \bar{a} = 2a_0$,

$$a(a + \bar{a}) = a^2 + a \cdot \bar{a} = 2a_0 a. \tag{23}$$

This result, which expresses a^2 in terms of a and scalars, can be used iteratively to find a^n, but it can also be used as a minimum polynomial equation to find scalar eigenvalues α and eigenelements x of a. Substituting

$$ax = \alpha x \tag{24}$$

into (23) we obtain

$$(\alpha^2 - 2a_0\alpha + a \cdot \bar{a})x = 0 \tag{25}$$

which gives eigenvalues

$$\alpha_{\pm} = a_0 \pm (\boldsymbol{a}^2)^{1/2}. \tag{26}$$

As long as $\boldsymbol{a}^2 \neq 0$, there are two distinct eigenvalues and the corresponding eigenelements x must satisfy (24), which may be rewritten

$$\hat{a}x = \pm x \tag{27}$$

where $\hat{a}$ is the unit vector $\hat{a} = \boldsymbol{a}/(\boldsymbol{a}^2)^{1/2}$. Multiplying equation (27) from the right by $\bar{x}$ we obtain

$$(1 \mp \hat{a})x\bar{x} = 0 \tag{28}$$

which proves that the scalar $x\bar{x} = 0$, i.e. that all eigenelements x must be null. The general solution x to (27) can be written

$$x = P_{\pm}y \tag{29}$$

where y is an arbitrary element of $\mathscr{P}$ and $P_{\pm}$ is the projector (the 'eigenprojector')

$$P_{\pm} = \tfrac{1}{2}(1 \pm \hat{a}) \tag{30}$$

which obeys

$$P_+ + P_- = 1 \qquad P_+P_- = P_-P_+ = 0 \qquad P_{\pm}^2 = P_{\pm}. \tag{31}$$

Note that, although we refer to $P_{\pm}$ as a projector, it is real iff $\hat{a}$ is real. The general solution (29) can always be written as a linear combination of $P_{\pm}$ and

$$P_{\pm}\hat{\xi} = \tfrac{1}{2}(\hat{\xi} \pm \mathrm{i}\hat{\eta}) = \hat{\xi}P_{\mp}$$

where $\hat{\xi}$ is any real unit vector perpendicular to $\hat{a}$ and $\hat{\eta} = \hat{a} \times \hat{\xi}$.

With the eigenprojectors $P_{\pm}$ and their eigenvalues $\alpha_{\pm}$, any power of a can be expanded

$$a^n = \alpha_+^n P_+ + \alpha_-^n P_- \tag{32}$$

and any analytic function $f(x)$ is seen by power series expansion to be defined when the argument is an element $a \in \mathscr{P}$ with $\boldsymbol{a}^2 \neq 0$ by

$$\begin{aligned} f(a) &= f(\alpha_+)P_+ + f(\alpha_-)P_- \\ &= f_+ + f_-\hat{a} \end{aligned} \tag{33}$$

where

$$f_{\pm} \equiv \tfrac{1}{2}[f(\alpha_+) \pm f(\alpha_-)].$$

The decomposition (33) is fully analogous to the secular decomposition useful for evaluating functions of diagonalisable matrices. It may even be used when $f(x)$ has no power series expansion. For example, consider the square root function. Noting that $\alpha_+ \alpha_- = a\bar{a}$, one easily verifies that

$$\begin{aligned} a^{1/2} &= \tfrac{1}{2}(\alpha_+^{1/2} + \alpha_-^{1/2}) + \tfrac{1}{2}\hat{a}(\alpha_+^{1/2} - \alpha_-^{1/2}) \\ &= \frac{1}{\sqrt{2}}\{[a_0 + (a\bar{a})^{1/2}]^{1/2} + \hat{a}[a_0 - (a\bar{a})^{1/2}]^{1/2}\} \\ &= \frac{a + (a\bar{a})^{1/2}}{2[a_0 + (a\bar{a})^{1/2}]^{1/2}} \end{aligned} \tag{34}$$

which can also be directly derived from (23).

If $\boldsymbol{a}^2=0$, a has only one eigenvalue, namely $1\cdot a=a_0$. If $\boldsymbol{a}=0$, a is a scalar and every element of $\mathscr{P}$ is an eigenelement. However, $\boldsymbol{a}$ may also be a complex vector proportional to $\hat{\xi}+\mathrm{i}\hat{\eta}$ where $\hat{\xi}$ and $\hat{\eta}$ are any perpendicular real unit vectors. It is easily seen that any linear combination of $\boldsymbol{a}$ and the projector

$$P=\tfrac{1}{2}(1+\hat{\xi}\times\hat{\eta}) \tag{35}$$

is an eigenelement, and that any differentiable function of a can be expanded about a_0 to give

$$f(a)=f(a_0)+\boldsymbol{a}f'(a_0) \tag{36}$$

where the prime indicates differentiation with respect to its argument.

4. 4-vectors, 6-vectors, and restricted Lorentz transformations

Minkowski-space 4-vectors take the form of real elements of $\mathscr{P}$: the zeroth or time component is added as a real scalar to the 3-space vector part. Some examples are

position	$x=t+\boldsymbol{x}$
momentum	$p=E+\boldsymbol{p}$
vector potential	$A=\phi+\boldsymbol{A}$
current density	$J=\rho+\boldsymbol{J}$
differential operator	$\partial=\partial/\partial t-\boldsymbol{\nabla}$.

If we think of these as being 4-vectors in their contravariant form, the corresponding covariant forms are their spatial reversals. The modulus of a 4-vector is seen to be its Minkowski-space square norm.

The spatial reflection (12) is easily extended to 4-vectors:

$$x\to\hat{n}\bar{x}^{+}\hat{n} \tag{37}$$

and since the scalar part t commutes with all elements, the rotation transformation (15) for x has the same form as for the vector:

$$x\to\exp(-\mathrm{i}\boldsymbol{\theta}/2)x\exp(\mathrm{i}\boldsymbol{\theta}/2). \tag{38}$$

The extension to the group $L_+^\uparrow$ of restricted Lorentz transformations is achieved if we allow the exponential parameter to be complex:

$$x\to LxL^{+} \tag{39}$$

where the transformation L is a unimodular element of $\mathscr{P}$:

$$L=\exp(\boldsymbol{w}/2-\mathrm{i}\boldsymbol{\theta}/2) \tag{40}$$

whose representations form the six-parameter group SL(2, C) of unimodular complex 2×2 matrices. If $\boldsymbol{w}=0$, L is unitary ($L^{+}=\bar{L}$) and the transformation is a pure rotation. If $\boldsymbol{\theta}=0$, L is real ($L=L^{+}$) and the transformation is a pure boost (velocity transformation).

Consider, for example, the 4-position $r = t + \boldsymbol{r}$ of a particle transformed from its rest frame. Since the rest frame 4-position is simply the proper time τ, a scalar, the boost transformation (39) gives

$$r = L\tau L^{+} = LL^{+}\tau = u\tau \tag{41}$$

where $u = \mathrm{d}r/\mathrm{d}\tau$ is the 4-velocity. In the case of a pure boost, $L = L^{+} = u^{1/2}$ with

$$u = \gamma + \boldsymbol{u} = \mathrm{e}^{\boldsymbol{w}} = \cosh w + \hat{\boldsymbol{w}} \sinh w. \tag{42}$$

Lorentz transformations of products of 4-vectors are easily derived from (39). Products in which 4-vectors alternate with spatial reversals of 4-vectors have especially simple transformations. Thus, for example, the product

$$\begin{aligned} r\bar{p} &= (t + \boldsymbol{r})(E - \boldsymbol{p}) \\ &= (tE - \boldsymbol{r} \cdot \boldsymbol{p}) + \boldsymbol{r}E - t\boldsymbol{p} - \mathrm{i}\boldsymbol{r} \times \boldsymbol{p} \end{aligned} \tag{43}$$

transforms as

$$r\bar{p} \rightarrow LrL^{+}\overline{LpL^{+}} = Lr\bar{p}\bar{L}. \tag{44}$$

The scalar part (in parentheses on the RHS of (43))

$$l \cdot r\bar{p} = r \cdot \bar{p} \equiv (r\bar{p} + p\bar{r})/2 = (\bar{r}p + \bar{p}r)/2 \tag{45}$$

is Lorentz invariant. It is, of course, the Minkowski-space scalar product of the 4-vectors r and p. The rest of $r\bar{p}$ is a vector plus a pseudovector, which together is called a 6-vector. (It may be recognised that the 6-vector part of $r\bar{p}$ is the generator of boosts and rotations in quantum theory. Indeed, the six parameters $\boldsymbol{w}/2 - \mathrm{i}\boldsymbol{\theta}/2$ of a restricted Lorentz transformation are the components of a 6-vector.)

All alternating products of 4-vectors and spatial reversals thereof transform like 4-vectors ((39) or its spatial reversal) if there are an odd number of terms, or like 6-vectors ((44) or its complex conjugate) if there are an even number. An important 6-vector is the electromagnetic field $\boldsymbol{F} = \boldsymbol{E} + \mathrm{i}\boldsymbol{B}$, which in a Lorentz gauge $(\bar{\partial} \cdot A = 0)$ is simply

$$\boldsymbol{F} = \boldsymbol{E} + \mathrm{i}\boldsymbol{B} = \partial\bar{A}. \tag{46}$$

The gauge-independent expression in terms of the 4-potential A is

$$\boldsymbol{F} = (\partial\bar{A} - A\bar{\partial})/2 \tag{46a}$$

where the differential operator can operate to the left as well as to the right. A pure boost of $\boldsymbol{F}$ gives

$$\boldsymbol{F} \rightarrow \exp(\boldsymbol{w}/2)\boldsymbol{F}\exp(-\boldsymbol{w}/2) = \boldsymbol{F}_{\parallel} + u\boldsymbol{F}_{\perp} = \boldsymbol{F}_{\parallel} + \gamma\boldsymbol{F}_{\perp} + \mathrm{i}\boldsymbol{u} \times \boldsymbol{F} \tag{47}$$

where $\boldsymbol{F}_{\parallel} = \boldsymbol{F} - \boldsymbol{F}_{\perp} = \boldsymbol{F} \cdot \hat{\boldsymbol{w}}\hat{\boldsymbol{w}}$. Like $\boldsymbol{F}$, 6-vectors in $\mathscr{P}$ are equivalent to antisymmetric second-rank tensors in Minkowski space, but their coordinate-free representation in $\mathscr{P}$ as complex vectors is more compact and more intuitive. The duality rotation of the fields (Jackson 1975, p 252) is simply

$$\boldsymbol{F} \rightarrow \exp(-i\phi)\boldsymbol{F}. \tag{48}$$

Although the elements of $\mathscr{P}$ are combinations of scalars and vectors, higher-rank covariant quantities often appear naturally in contracted form. For example, the 4-vector quantity

$$\boldsymbol{F}a\boldsymbol{F}^{+} \tag{49}$$

where $\boldsymbol{F}$ is the electromagnetic field (46) and $a = a_0 + \boldsymbol{a}$ is an arbitrary 4-vector, is easily seen to be

$$\boldsymbol{F}a\boldsymbol{F}^{+} = 8\pi a_0(U + \boldsymbol{S}) + 8\pi \boldsymbol{a}\cdot(\overleftrightarrow{T} - \boldsymbol{S}) \tag{50}$$

when U is the energy density, $\boldsymbol{S}$ is the momentum density or Poynting vector, and $\overleftrightarrow{T}$ is the Maxwell stress dyad:

$$8\pi U = E^2 + B^2 \tag{51a}$$

$$4\pi \boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{B} \tag{51b}$$

$$4\pi\overleftrightarrow{T} = \overleftrightarrow{\boldsymbol{EE}} + \overleftrightarrow{\boldsymbol{BB}} - 4\pi U\overleftrightarrow{1}, \tag{51c}$$

expressed here in Gaussian units with $c = 1$. One sees immediately that $\boldsymbol{F}a\boldsymbol{F}^{+}$ and hence $U, \boldsymbol{S}$ and $\overleftrightarrow{T}$ are invariant under duality rotations (48).

5. Minkowski-space components

An important advantage of the Pauli algebra is that calculations require no vector or matrix components. Nevertheless, elements of $\mathscr{P}$ can be expanded in a Minkowski-space basis if desired, as for example when comparing results in $\mathscr{P}$ with those from other approaches.

We define a basis $\{\sigma_\mu\}_{\mu=0,1,2,3}$ of elements in $\mathscr{P}$ by

$$\begin{aligned} &\sigma_0 = \bar{\sigma}_0 = 1 \qquad && \sigma_1 = -\bar{\sigma}_1 = \hat{x} \\ &\sigma_2 = -\bar{\sigma}_2 = \hat{y} \qquad && \sigma_3 = -\bar{\sigma}_3 = \hat{z}. \end{aligned} \tag{52}$$

From the definition (17) of a scalar product, we see

$$\sigma_\mu\cdot\sigma_\nu = \bar{\sigma}_\mu\cdot\bar{\sigma}_\nu = g_\mu^\nu = \begin{cases} 1 & \mu=\nu \\ 0 & \mu\neq\nu. \end{cases} \tag{53}$$

The Lorentz scalars

$$\sigma_\mu\cdot\bar{\sigma}_\nu = \bar{\sigma}_\mu\cdot\sigma_\nu = g_{\mu\nu} = g^{\mu\nu} = \begin{cases} 1 & \mu=\nu=0 \\ -1 & \mu=\nu=1,2,3 \\ 0 & \mu\neq\nu \end{cases} \tag{54}$$

are elements of the Minkowski-space metric tensor. An arbitrary element a of $\mathscr{P}$ can be expanded in the basis (repeated indices are to be summed)

$$a = a^\mu\sigma_\mu = a_\nu\bar{\sigma}_\nu \tag{55}$$

where the contravariant components a^μ and covariant components a_ν are found from (53):

$$a^\mu = a\cdot\sigma_\mu \qquad a_\nu = a\cdot\bar{\sigma}_\nu. \tag{56}$$

As a simple example, the Lorentz-scalar product of two 4-vectors a and b of $\mathscr{P}$ is

$$a \cdot \bar{b} = a^{\mu}\sigma_{\mu} \cdot \bar{\sigma}_{\nu} b^{\nu} = a^{\mu} g_{\mu\nu} b^{\nu}$$
$$= a^{\mu} b_{\mu} = a_{\nu} b^{\nu}. \tag{57}$$

Other products of 4-vectors may be similarly expanded. Consider the 6-vector

$$f = -\bar{f} = \tfrac{1}{2}(a\bar{b} - b\bar{a}) = a\bar{b} - a \cdot \bar{b}. \tag{58}$$

By expanding the 4-vectors a, $\bar{a}$, b and $\bar{b}$ in basis elements, as in (55), we obtain

$$f = \tfrac{1}{2}(a^{\mu} b^{\nu} - a^{\nu} b^{\mu})\sigma_{\mu}\bar{\sigma}_{\nu} \equiv \tfrac{1}{2} f^{\mu\nu} \tau_{\mu\nu} \tag{59}$$

where

$$f^{\mu\nu} = -f^{\nu\mu} = (a^{\mu} b^{\nu} - a^{\nu} b^{\mu}) \tag{60}$$

is the antisymmetric second-rank Minkowski-space tensor associated with f and

$$\tau_{\mu\nu} = \tfrac{1}{2}(\sigma_{\mu}\bar{\sigma}_{\nu} - \sigma_{\nu}\bar{\sigma}_{\mu}) = \sigma_{\mu}\bar{\sigma}_{\nu} - g_{\mu\nu} \tag{61}$$

is an antisymmetric matrix of basis elements. Multiplication in $\mathscr{P}$ gives directly

$$\tau_{\mu\nu} = (g_{\mu\alpha} g_{\nu 0} - g_{\nu\alpha} g_{\mu 0} + \mathrm{i}\varepsilon_{\mu\nu\alpha 0})\bar{\sigma}_{\alpha} \tag{62}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the fully antisymmetric fourth-rank tensor equal to $+1$ (-1) whenever the indices $\mu\nu\alpha\beta$ are an even (odd) permutation of 1 2 3 0. The relation (59) between tensor elements $f^{\mu\nu}$ and 6-vector components of f is easily seen when $\tau_{\mu\nu}$ (62) is written in matrix form:

$$\tau_{\mu\nu} = \begin{pmatrix} 0 & -\hat{x} & -\hat{y} & -\hat{z} \\ \hat{x} & 0 & -\mathrm{i}\hat{z} & \mathrm{i}\hat{y} \\ \hat{y} & \mathrm{i}\hat{z} & 0 & -\mathrm{i}\hat{x} \\ \hat{z} & -\mathrm{i}\hat{y} & \mathrm{i}\hat{x} & 0 \end{pmatrix} \tag{63}$$

or equivalently, when the expansion (62) is substituted directly into (59),

$$f = (f^{k0} - \mathrm{i}\varepsilon_{ljk} f^{lj})\sigma_k \tag{64}$$

where the indices j, k and l are summed over the values 1, 2 and 3, and

$$\varepsilon_{ljk} = (\sigma_l \times \sigma_j) \cdot \sigma_k = \varepsilon_{ljk0} \tag{65}$$

is the usual Levi-Cevita symbol of 3-space.

The corresponding relations for covariant components are found by replacing all the 4-vector factors by their spatial reversals in $\mathscr{P}$, without changing their order. Thus for 6-vectors formed from bilinear factors of real 4-vectors, the relation (59) becomes

$$\bar{f}^{+} = \tfrac{1}{2}(\bar{a}b - \bar{b}a) = \tfrac{1}{2} f_{\mu\nu}\tau_{\mu\nu} \tag{66}$$

where it has been assumed that $a = a^{+}$ and $b = b^{+}$.

Dual tensors

$$f^{*\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} f_{\alpha\beta} = \tfrac{1}{2} f_{\alpha\beta}\varepsilon^{\alpha\beta\mu\nu} \tag{67}$$

with $\varepsilon^{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}$ correspond simply to $-\mathrm{i}f$, as may be seen from

$$\tau^{\mu\nu} \equiv \tfrac{1}{2}(\bar{\sigma}_{\mu}\sigma_{\nu} - \bar{\sigma}_{\nu}\sigma_{\mu}) = \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\alpha\beta}\tau_{\alpha\beta} \tag{68}$$

since then (see (59))

$$-\mathrm{i}f = -\tfrac{1}{2}\mathrm{i} f^{\mu\nu}\tau_{\mu\nu} = -\tfrac{1}{2}\mathrm{i} f_{\mu\nu}\tau^{\mu\nu}$$
$$= \tfrac{1}{2}(\tfrac{1}{2} f_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta})\tau_{\alpha\beta} = \tfrac{1}{2} f^{*\alpha\beta}\tau_{\alpha\beta}. \tag{69}$$

Higher-order products of 4-vectors in $\mathcal{P}$ can be related to Minkowski-space tensors by similarly expanding each 4-vector in basis elements σ_μ and re-expressing products of basis elements as linear combinations of the σ_μ themselves, through iterative application of (61) and (62). However, because operations with component-free elements of $\mathcal{P}$ are usually much simpler than manipulating Minkowski-space components, it is usually advantageous to complete the operations in $\mathcal{P}$ before expanding in components. For example, the Lorentz transformation of a 4-vector a,

$$a = a^\nu \sigma_\nu \to LaL^+ = a^\mu L\sigma_\mu L^+ \tag{70}$$

is easily used to find components $L^\nu{}_\mu$ of the transformation

$$a^\nu \to L^\nu{}_\mu a^\mu. \tag{71}$$

One finds

$$L^\nu{}_\mu = \sigma_\nu \cdot (L\sigma_\mu L^+) = \bar{\sigma}_\nu \cdot \overline{(L\sigma_\mu L^+)}. \tag{72}$$

Thus for a pure boost along the x axis to 4-velocity $u = \gamma + \boldsymbol{u} = LL^+$ (see (42)), since σ_0 and σ_1 commute with $\boldsymbol{u} = |\boldsymbol{u}|\hat{x}$ whereas σ_2 and σ_3 anticommute,

$$L\sigma_\mu L^+ = \begin{cases} LL^+\sigma_\mu = u\sigma_\mu & \mu = 0, 1 \\ L\bar{L}^+\sigma_\mu = \sigma_\mu & \mu = 2, 3 \end{cases} \tag{73a}$$

$$= \begin{cases} \gamma + \boldsymbol{u} & \mu = 0 \\ |\boldsymbol{u}| + \gamma\hat{x} & \mu = 1 \\ \sigma_\mu & \mu = 2, 3 \end{cases} \tag{73b}$$

and the 4×4 matrix $(L^\nu{}_\mu)$ is

$$(L^\nu{}_\mu) = \begin{pmatrix} \gamma & |\boldsymbol{u}| & 0 & 0 \\ |\boldsymbol{u}| & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{74}$$

The same matrix elements appear in Lorentz transformations of all 4-vector products. For example, 6-vectors transform according to (44):

$$\begin{aligned} f &= \tfrac{1}{2} f^{\alpha\beta}\sigma_\alpha\bar{\sigma}_\beta \to Lf\bar{L} = \tfrac{1}{2} f^{\mu\nu} L\sigma_\mu\bar{\sigma}_\nu\bar{L} \\ &= \tfrac{1}{2} f^{\mu\nu}(L\sigma_\mu L^+)\overline{(L\sigma_\nu L^+)} \\ &= \tfrac{1}{2} L^\alpha{}_\mu L^\beta{}_\nu f^{\mu\nu}\sigma_\alpha\bar{\sigma}_\beta \end{aligned} \tag{75}$$

and thus in terms of Minkowski-space tensors

$$f^{\alpha\beta} \to L^\alpha{}_\mu L^\beta{}_\nu f^{\mu\nu}. \tag{76}$$

Higher-rank tensors which appear in $\mathcal{P}$ in contracted form may be related to their tensor counterparts by analogous expansions in the basis elements σ_μ. For example, the 4-vector $\boldsymbol{F}a\boldsymbol{F}^+$ (50) is seen to arise from the contraction of the symmetrical electromagnetic stress tensor $T^{\mu\nu}$ (Jackson 1975, p 605) with the arbitrary 4-vector a^ν:

$$\boldsymbol{F}a\boldsymbol{F}^+ = 8\pi\sigma_\mu T^\mu{}_\nu a^\nu. \tag{77}$$

However, it is usually simpler to manipulate the quantities in $\mathcal{P}$. If $a = 1$ in (77) we obtain the real element

$$G \equiv U + \boldsymbol{S} = \frac{1}{8\pi} \boldsymbol{F}\boldsymbol{F}^+ = \sigma_\mu T^{\mu 0} \tag{78}$$

which gives the 4-momentum density. Although the transformation properties of G are not simple, its norm

$$G\bar{G} = U^2 - S^2 = T_{\mu 0}T^{\mu 0} \tag{79}$$

is immediately seen in $\mathscr{P}$ (but not from the Minkowski-tensor expression) to be a Lorentz invariant.

6. Conclusions

Clifford algebras are known to offer compact component-free approaches to relativistic physics which emphasise the geometrical significance of the objects and their manipulations. What is surprising, and apparently often overlooked, is that the Pauli algebra, although it is the Clifford algebra for Euclidean 3-space, naturally gives rise to elements which are equivalent to the 4-vectors and higher-order tensors of Minkowski space and to their contractions in the Minkowski metric. The success of the Pauli algebra in describing relativistic phenomena suggests an alternative to the usual view of spacetime. Instead of working with a four-dimensional space, whose only restriction on homogeneity enters through a metric tensor which treats the time coordinates distinctly from the three space coordinates, the Pauli algebra deals explicitly with a three-dimensional space, and time enters not as the new dimension of an enlarged vector space, but rather as the scalar part of an element in the Clifford algebra of the Euclidian 3-space. The Pauli algebra approach thus sharpens the distinction between the time part and the three space components of what in Minkowski space is a 4-vector. The distinction is manifested in the invariance of the time part under transformation of the rotation group SU(2). Time and space are 'unified' in elements of $\mathscr{P}$ and are mixed by other Lorentz transformations which are defined to leave invariant these bilinear products of elements in $\mathscr{P}$ which are pure scalars. By introducing simple symbols to denote elements of $\mathscr{P}$, we obtain a covariant formulation of special relativity.

In this paper, we have presented the Pauli algebra in a formulation convenient for applications in relativistic physics. In contrast to higher Clifford algebras, such as the Dirac algebra, multiplication in $\mathscr{P}$ is easily expressed in terms of familiar dot and cross products of 3-vectors. In contrast to the usual Minkowski-tensor approach, tensor components and their implicit reference to a system of coordinates are not necessary, and the geometric and physical significance of the tensors is usually more obvious. Unimodular elements have been seen to effect Lorentz transformations.

Elements of $\mathscr{P}$ may be classified according to how they transform under restricted Lorentz transformations. In particular, we have distinguished elements which transform like 4-vectors of Minkowski space from products which transform like antisymmetric second-rank Minkowski-space tensors (one '6-vector') (see (39) and (44)). It may sometimes be useful to introduce elements with other transformation behaviour. For example, a *spinor element* [rank $(1,0)$] ξ of $\mathscr{P}$ may be defined by the behaviour

$$\xi \xrightarrow{L} L\xi. \tag{80}$$

A Lorentz transformation $L_1 = \exp(\boldsymbol{w}_1/2 - \mathrm{i}\boldsymbol{\theta}_1/2)$ may itself be considered a spinor element if we take the result of a Lorentz transformation L_2 on L_1 to be the combined transformation L_2L_1 (Hestenes 1966). (The alternate view, that the transformed L_1 is L_1 as viewed in the frame from which the lab frame is reached by L_2, gives a 6-vector transformation behaviour for L_1 as well as for $\boldsymbol{w}_1/2 - \mathrm{i}\boldsymbol{\theta}_1/2$. A spinor behaviour (80) of L_1 is of course consistent with the 4-vector transformation of the product $u_1 = L_1 L_1^+$ (see (41)).

If η is a rank $(1, 0)$ spinor element of $\mathscr{P}$, the conjugate of its spatial reverse transforms as a rank $(0, 1)$ spinor:

$$\bar{\eta}^+ \xrightarrow{L} \bar{L}^+ \bar{\eta}^+. \tag{81}$$

Such spinors are involved in the Dirac equation. In $\mathscr{P}$ the Dirac equation follows from the inequivalence of the quantum operators $p\bar{p}$ and $\bar{p}p$ in the presence of an external 4-potential A where (with $\hbar = 1$)

$$p = \mathrm{i}\partial - qA \tag{82}$$

and q is the charge. The separate covariant equations

$$p\bar{p}\xi = m^2 \xi \tag{83a}$$

$$\bar{p}p\bar{\eta}^+ = m^2 \bar{\eta}^+ \tag{83b}$$

lead, with an added constraint on the relative norms of the spinor wavefunctions ξ and $\bar{\eta}^+$, to the equivalent first-order coupled equations (Baylis 1980)

$$\bar{p}\xi = m\bar{\eta}^+ \tag{84a}$$

$$p\bar{\eta}^+ = m\xi \tag{84b}$$

which are easily written as a single equation for the bispinors:

$$\psi = \xi P_+ + \bar{\eta}^+ P_- \tag{85}$$

where $P_\pm$ are the real projectors (30)

$$P_\pm = \tfrac{1}{2}(1 \pm \hat{z}). \tag{86}$$

One finds

$$p\psi P_- \pm \bar{p}\psi P_+ = m\psi\hat{x} \tag{87}$$

as the Dirac equation in $\mathscr{P}$ in the spinor representation. The bispinor ψ transforms under Lorentz transformations according to

$$\psi \to L\psi P_+ + \bar{L}^+ \psi P_-. \tag{88}$$

From (85) and (86), the matrix representation of the bispinors in the spinor representation is a 2×2 matrix where the first column is a two-element rank $(1, 0)$ spinor transforming like ξ (80), and where the second column is a rank $(0, 1)$ spinor transforming like $\bar{\eta}^+$ (81). Under a rotation, both parts of ψ transform the same way:

$$\psi \to R\psi. \tag{89}$$

Other representations are obtained by multiplying ψ on the right with a unitary element of $\mathscr{P}$, which represents a rotation 'in particle-antiparticle space' (in contrast to (89) which may be described as a rotation in 'spin space'):

$$\psi \to \psi R. \tag{90}$$

The standard representation, for example, is obtained by the 180° rotation which interchanges $\hat{x} \leftrightarrow \hat{z}$.

Much work remains to exploit fully the power of the Pauli algebra approach to relativistic physics. We have shown that the approach is not limited to the 4- and 6-vectors of classical physics. However, in the following paper (Baylis and Jones 1989) we will concentrate on problems in classical electrodynamics.

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